Problem Set 9 due May 14, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix}$.

(1) Compute the SVD of A, and the pseudo-inverse A^+. *(15 points)*

(2) Compute the vector $\boldsymbol{v}^+$ defined by formula (263) in the lecture notes, which will have the property that $\boldsymbol{p} := A\boldsymbol{v}^+$ is as close to $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as possible. *(5 points)*

(3) Compute all solutions to $A\boldsymbol{v} = \boldsymbol{p}$ and prove that $\boldsymbol{v}^+$ is the shortest one. *(10 points)*

Problem 2:

(1) Compute the sixth roots of unity (i.e. the complex numbers z such that $z^6 = 1$) in both Cartesian (i.e. $a + bi$) and polar (i.e. $re^{i\theta}$) form. Draw them all on a picture of the plane.

(10 points)

(2) Prove the double angle and triple angle formulas:

$$\cos(2\theta) = 2(\cos\theta)^2 - 1 \qquad \text{and} \qquad \cos(3\theta) = 4(\cos\theta)^3 - 3\cos\theta \tag{1}$$

by the following logic:

- think of $\cos\theta$ as the real part of the complex number $z = e^{i\theta} = a + bi$ where $a = \cos\theta$, $b = \sin\theta$
- then compute z^2 (respectively z^3) first in polar form, and
- finally compute z^2 (respectively z^3) in Cartesian form.

By equating the results in the last two bullets, you should obtain (1). *(10 points)*

Problem 3:

Suppose we have a pair of trick coins which we toss one after the other, and they behave as follows. The first coin is fair, but if the first coin shows heads then the second coin will automatically show head as well, while if the first coin shows tails then the second coin is fair.

(1) Consider the random vector:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where X_1 (respectively X_2) is 0 or 1 depending on whether the first (respectively second) coin shows heads or tails. Compute the mean $\boldsymbol{\mu}$ and covariance matrix K of $\boldsymbol{X}$. *(10 points)*

(2) Diagonalize the covariance matrix $K = QDQ^T$ where Q is orthogonal and D is diagonal.

(10 points)

(3) Do principal component analysis to find two linear combinations of X_1 and X_2 which are uncorrelated (i.e. have covariance 0). Find the variances of these linear combinations. *(10 points)*

Problem 4:

(1) Prove that the total probability of a normal distribution is 1, by the following algorithm:

- Let $\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$. Fill in the blank: the goal is to show that ____________
- Prove that $\alpha^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$.
- Prove that $\alpha^2 = 1$ by converting the double integral to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Recall that the differential changes according to $dxdy = rdrd\theta$. Conclude that $\alpha = 1$.

Hint: The last bullet requires you to perform a little integration trick. *(10 points)*

(2) Let $\boldsymbol{X}$ be a random vector with mean $\boldsymbol{\mu}$. Show that the covariance matrix, which is defined by:

$$E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X}^T - \boldsymbol{\mu}^T)]$$

can also be given by the formula $E[\boldsymbol{X}\boldsymbol{X}^T] - \boldsymbol{\mu}\boldsymbol{\mu}^T$ (this is handy if you're ever trying to compute a covariance matrix, and don't want to deal with all those $\boldsymbol{\mu}$'s being subtracted). *(10 points)*